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VARIATIONS ON A THEME: THE POSITIVE COMMUTATOR IN SPECTRAL GEOMETRY

HANS CHRISTIANSON

ABSTRACT. This work is a summary and expansion on a talk given by the author in December, 2016 at the conference "Spectral and Scattering Theory and Related Topics" at RIMS Kyoto. The work in this expository note was done with collaborators J. Wunsch and J. Metcalfe.

1. INTRODUCTION

Our motivation for the results in this paper is to better understand properties of solutions to dispersive equations in different geometric settings. Our techniques are to study the underlying stationary problem and try to gain information about how the local and global geometry affects resolvent estimates, resonances, and (in compact settings) eigenfunctions and eigenvalues. These questions all fall into the category of *spectral geometry*, which is, as it sounds, the study of spectral theory in different geometric situations.

In these notes, we focus on the "positive commutator" methods developed and refined by many authors, beginning with Morawetz [17]. We will first focus on the situation of the Schrödinger equation on a warped product manifold, where it is most easy to understand trapped sets. An easy way to begin seeing these effects is to study the local smoothing effect. This is in Section 2. First we discuss a little history of the use of commutator arguments in the study of partial differential equations.

1.1. History.

It has been known for a long time that if one writes the Laplacian on $\mathbb{R}^n$ in polar coordinates (r, ω), the commutator with the radial vector field reproduces the Laplacian:

$$[-\Delta, r\partial_r] = -2\Delta.$$

As the symbol $|\xi|^2$* of the Laplacian generates the straight line geodesic flow on $\mathbb{R}^n$, this identity expresses that the radial vector "increases" along straight lines going to spatial infinity. We refer to this commutator as the "grand-daddy" commutator.

As mentioned above, Morawetz used this identity in [17] to study localized energy decay for the wave equation. These are estimates that express that if one looks at energy locally in space, then it "escapes" to infinity, or decays in time. Due to the logarithmic singularity in the resolvent expansion in even dimensions, the explicit time dependence of this decay is complicated, but it turns out for applications to nonlinear wave equations, an energy which is intergrated in time is actually more useful anyway. In other words, under suitable assumptions, the energy of solutions to the wave equation measured in compact regions is integrable in time (as opposed to constant in time for the energy measured in all of $\mathbb{R}^n$.)

*Here ξ denotes the Fourier dual variable of the spatial variable x.

One of the next major uses of the positive commutator idea was in Hörmander's results on propagation of singularities for linear PDEs [13]. Here, this idea is generalized to reasonable self-adjoint operators whose associated classical flow is non-trapping. It says that a singularity of a solution to a PDE that initiates at a particular place in phase space actually propagates to be singular on the entire bicharacteristic ray through that point. This was primarily a result about stationary elliptic equations, but can be applied in many different geometric situations as well (for example, it can be used to reproduce the flow of wave packets for the wave equation). The proof is in similar spirt to some of the proofs in this paper, using that one can cook up quantities that increase along the Hamiltonian flow of a real-valued symbol, at least away from places where the flow is stationary. In the vocabulary of this note, stationary or periodic invariant sets are parts of a "trapped" set, which is a classical invariant set which does not escape to infinity under the classical flow.

In the 1980's, the use of commutators in dispersive equations to understand solutions to non-linear but integrable equations [16], more general linear dispersive equations [10, 19].

2. Local smoothing effect

We begin with a simple proof of the local smoothing property for solutions to the Schrödinger equation in $\mathbb{R}^2$. This is a good starting point for this discussion, as the approach of the author and J. Wunsch in [9] begins with this on a manifold to detect where defects in local smoothing occur.

2.1. **The smoothing estimate on Euclidean space.** In this section we write out the standard positive commutator proof of local smoothing for the Schrödinger equation in polar coordinates. We then try to mimic the proof in the case of degenerate hyperbolic orbits in the next subsection to see where the proof fails.

In polar coordinates, the homogeneous Schrödinger equation on $\mathbb{R}_t \times \mathbb{R}^2$ is

$$\begin{cases} (D_t - \partial_r^2 - r^{-1}\partial_r - r^{-2}\partial_\theta^2)u = 0, \\ u|_{t=0} = u_0; \end{cases}$$

we will of course write

$$\Delta = \partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2.$$

We recall that in polar coordinates the radial, or scaling, vector field is $x \cdot \partial_x = r\partial_r$. By scaling, we immediately compute

$$[r\partial_r, \Delta] = 2\Delta;$$

however, as $r\partial_r$ is not a bounded map between Sobolev spaces, we change the weight and employ the commutant $B = r\langle r\rangle^{-1}\partial_r$. The function $a(r) = r\langle r\rangle^{-1}$ is non-negative and bounded, and satisfies $a'(r) = \langle r\rangle^{-3}$. Thus, we compute

$$\begin{aligned} [B, \Delta] &= 2a'\partial_r^2 + (a'' + a'r^{-1} + ar^{-2})\partial_r + 2ar^{-3}\partial_\theta^2 \\ &= 2\langle r\rangle^{-3}\partial_r^2 + 2\langle r\rangle^{-1} r^{-2}\partial_\theta^2 + O(r^{-1}\langle r\rangle^{-1})\partial_r. \end{aligned} \tag{2.1}$$

Using the Schrödinger equation, we write

$$\begin{aligned}
0 =& 2i \operatorname{Im} \int_0^T \langle B(D_t - \Delta)u, u\rangle \, dt \\
=& \int_0^T \left\langle B(D_t - \partial_r^2 - r^{-1}\partial_r - r^{-2}\partial_\theta^2)u, u\right\rangle \\
& - \left\langle u, B(D_t - \partial_r^2 - r^{-1}\partial_r - r^{-2}\partial_\theta^2)u\right\rangle dt \\
=& \int_0^T \langle [B, (-\partial_r^2 - r^{-1}\partial_r - r^{-2}\partial_\theta^2)]u, u\rangle \, dt + i \left.\langle Bu, u\rangle\right|_0^T .
\end{aligned}$$

The last term is bounded using energy estimates by

$$\left| \left.\langle Bu, u\rangle\right|_0^T \right| \leqslant \|u_0\|_{H^{1/2}}^2 .$$

Rearranging, we thus obtain

$$\int_0^T \langle [B, \Delta]u, u\rangle \, dt \leqslant C_T \|u_0\|_{H^{1/2}}^2 .$$

Employing (2.1) and integrating by parts thus yields

$$\int_0^T \left\| \langle r\rangle^{-3/2} \partial_r u \right\|^2 + \left\| \langle r\rangle^{-1/2} r^{-1} \partial_\theta u \right\|^2 dt \leqslant C_T \|u_0\|_{H^{1/2}}^2 .$$

where we have absorbed on the right the term involving $\int_0^T \langle \partial_r u, u\rangle \, dt$ as well as the similar error terms from commuting ∂_r with a multiplier. This is the local smoothing estimate on the manifold $\mathbb{R}^2$.

2.2. **Geometry with Trapping.** If we consider a semi-classical pseudodifferential operator $P(x, hD)$ with symbol $p(x, \xi)$, the propagation of singularities result of Hörmander [13] shows that singularities of solutions to the equation $P(x, hD)u = 0$ travel along the Hamiltonian flow of the symbol p:

$$\begin{cases} \dot{x} = p_\xi, \\ \dot{\xi} = -p_x. \end{cases}$$

If there are points (x_0, ξ_0) in phase space (here x_0 is the intial point and ξ_0 is the initial (co)direction of flow) such that the flow line emanating from (x_0, ξ_0) remains in a trapped set, we call these flow lines *trapped.*

A celebrated theorem of Doi [12] tells us that if one considers solutions to the Schrödinger equation on an asymptotically Euclidean manifold, then a perfect $1/2$ derivative local smoothing effect (such as in $\mathbb{R}^n$) holds if and only if the manifold is non-trapping. This tells us that interesting losses in regularity occur only in trapping situations. Hence understanding how local smoothing changes in different trapping situations becomes very interesting.

If the trapping is sufficiently unstable (non-degenerate hyperbolic trapping), there is a "trivial" loss of $\epsilon > 0$ derivatives from the $1/2$ derivative smoothing effect. This holds for boundary value problems [1, 5], sufficiently "thin" hyperbolic trapped sets [1, 3, 4, 11], and situations where there are some stable directions of the trapping [5]. In fact, with some care in definitions, the loss is only logarithmic. These examples include Ikawa's examples [1, 14, 15], a single periodic hyperbolic geodesic (with or without boundary reflections) [4], very general fractal trapped sets without boundary [3, 11, 18], and normally hyperbolic trapped sets [22]. That is, in all of these cases, the authors prove that for any $\epsilon > 0$, there exists a constant $C > 0$ such that

$$\int_0^T \| \langle x \rangle^{-1/2-\epsilon} e^{it\Delta} u_0 \|_{H^{1/2-\epsilon}}^2 dt \leqslant C \|u_0\|_{L^2}^2.$$

In this case, we call the loss due to trapping "trivial".

To contrast, if a manifold admits an elliptic trapped set (meaning stable under small perturbations of the initial point of the classical flow), the existence of resonances converging rapidly to the real axis and the existence of infinite order quasimodes prevents polynomial gain in regularity.

Along with J. Wunsch [9], the author studies a class of manifolds with only one periodic geodesic which is weakly hyperbolic, and prove a (sharp) local smoothing effect with loss that lies somewhere between the complete loss of an elliptic trapped set and the trivial loss of a strictly hyperbolic trapped set.

We consider the manifold $X = \mathbb{R}_x \times \mathbb{R}_\theta / 2\pi\mathbb{Z}$, equipped with a metric of the form

$$ds^2 = dx^2 + A^2(x) d\theta^2,$$

where $A \in \mathcal{C}^\infty$ is a smooth function, $A \geqslant \epsilon > 0$. From this metric, we get the volume form

$$d\mathrm{Vol} = A(x) dx d\theta,$$

and the Laplace-Beltrami operator acting on 0-forms

$$\Delta f = (\partial_x^2 + A^{-2}\partial_\theta^2 + A^{-1}A'\partial_x) f.$$

We observe that we can conjugate Δ by an isometry of metric spaces and separate variables so that spectral analysis of Δ is equivalent to a one-variable semiclassical problem with potential. That is, let $T : L^2(X, d\mathrm{Vol}) \to L^2(X, dx d\theta)$ be the isometry given by

$$Tu(x, \theta) = A^{1/2}(x) u(x, \theta).$$

Then $\widetilde{\Delta} = T \Delta T^{-1}$ is essentially self-adjoint on $L^2(X, dx d\theta)$ with mild assumptions on A (for example in this paper X has two ends which are short range perturbations of $\mathbb{R}^2$). A simple calculation gives

$$-\widetilde{\Delta} f = (-\partial_x^2 - A^{-2}(x)\partial_\theta^2 + V_1(x)) f,$$

where the potential

$$V_1(x) = \frac{1}{2} A'' A^{-1} - \frac{1}{4} (A')^2 A^{-2}.$$

If we now separate variables and write $\psi(x, \theta) = \sum_k \varphi_k(x) e^{ik\theta}$, we see that

$$(-\widetilde{\Delta} - \lambda^2)\psi = \sum_k e^{ik\theta} (P_k - \lambda^2) \varphi_k(x),$$

where

$$(P_k - \lambda^2)\varphi_k(x) = (-\frac{d^2}{dx^2} + k^2 A^{-2}(x) + V_1(x) - \lambda^2)\varphi_k(x).$$

Setting $h = k^{-1}$, we have the semiclassical operator

$$P(z, h)\varphi(x) = (-h^2 \frac{d^2}{dx^2} + V(x) - z)\varphi(x),$$

where the potential is

$$V(x) = A^{-2}(x) + h^2 V_1(x)$$

and the spectral parameter is $z = h^2 \lambda^2$.

In [9], we are primarily interested in the case $A(x) = (1+x^{2m})^{1/2m}$, $m \in \mathbb{Z}_+$. If $m \geqslant 2$, then X is asymptotically Euclidean (with two ends), and the subpotential $h^2 V_1(x)$ is seen to be lower order in both the semiclassical and the scattering sense. If $m = 1$, a trivial modification must be made to make the metric a short-range perturbation, but we completely ignore this issue here. The point is that for $m \geqslant 2$, the principal part of the potential $V(x)$ is $A^{-2}(x)$ which has a degenerate maximum at $x = 0$. The corresponding periodic geodesic $\gamma \subset X$ is *weakly* hyperbolic in the sense that it is unstable and isolated, but degenerate.

The main result in [9] is the following theorem, which says that for every $m \geqslant 2$, there is still some local smoothing, but with a polynomial loss depending on m.

Theorem 1 (Local Smoothing). *Suppose X is as above for $m \geqslant 2$, and assume u solves*

$$\begin{cases} (D_t - \Delta)u = 0 \text{ in } \mathbb{R} \times X, \\ u|_{t=0} = u_0 \in H^s \end{cases}$$

for some $s \geqslant m/(m+1)$. Then for any $T < \infty$, there exists a constant $C > 0$ such that

$$\int_0^T \| \langle x \rangle^{-3/2} u \|_{H^1(X)}^2 \, dt \leqslant C(\| \langle D_\theta \rangle^{m/(m+1)} u_0 \|_{L^2}^2 + \| \langle D_x \rangle^{1/2} u_0 \|_{L^2}^2).$$

Remark 2.1. Observe that there is no polynomial local smoothing effect in the limit $m \to \infty$. In [9] it is shown that Theorem 1 is sharp, and that in fact the estimate is saturated on a weak semiclassical time scale.

We are also able to prove, using the same techniques, a polynomial bound on the resolvent of the Laplacian in the same geometric setting.

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2.3. **Degenerate hyperbolic trapping.** In this section, we sketch the proof our main local smoothing estimate from [9].

2.4. **Returning to the commutator argument.** Let us begin by reproducing the positive commutator computation above for the degenerate case. Let $A(x) = (1+x^{2m})^{1/2m}$, the metric $ds^2 = dx^2 + A^2 d\theta^2$ as before, and conjugate the Laplacian to Euclidean space:

$$-\widetilde{\Delta} f = (-\partial_x^2 - A^{-2}(x)\partial_\theta^2 + V_1(x))f,$$

where the potential

$$V_1(x) = \frac{1}{2} A'' A^{-1} - \frac{1}{4}(A')^2 A^{-2}.$$

The following proposition is the statement of local smoothing for the conjugated equation, and evidently implies Theorem 1 by conjugating back.

Proposition 2.2. *Suppose $m \geqslant 2$ and u solves*

$$\begin{cases} (D_t - \widetilde{\Delta})u = 0, \\ u(0,x,\theta) = u_0. \end{cases} \tag{2.2}$$

Then for any $T < \infty$ there exists a constant $C > 0$ such that

$$\int_0^T (\| \langle x \rangle^{-1} \partial_x u\|_{L^2}^2 + \| \langle x \rangle^{-3/2} \partial_\theta u\|_{L^2}^2)\, dt \leqslant C(\| \langle D_\theta \rangle^{m/(m+1)} u_0\|_{L^2}^2 + \| \langle D_x \rangle^{1/2} u_0\|_{L^2}^2).$$

2.5. **Proof of Proposition 2.2.** Let us summarize briefly the strategy of the proof. Using a positive commutator argument similar to the previous section, we prove local smoothing except at the periodic orbit $\gamma = \{x = 0\}$. Moreover, solutions to (2.2) exhibit perfect local smoothing in the x direction and only lose smoothing in the directions tangential to γ (that is, only in the θ direction). Thus it suffices to prove local smoothing with a loss for θ derivatives, in a neighbourhood of $x = 0$. We separate variables in the θ direction (Fourier series decomposition) and prove estimates uniform in each Fourier mode. To do this, we further decompose, say, the kth Fourier mode into a low-frequency part where $|k| \leqslant |D_x|$ and a high-frequency part where $|D_x| \leqslant |k|$. The low frequency part is estimated using the positive commutator technique modulo a term which is localized to high-frequencies, so it suffices to estimate a solution cut off to high frequencies. For this, we introduce a semiclassical rescaling, and reduce the estimate to a cutoff semiclassical resolvent estimate, which implies local smoothing via [3, Theorem 1].

Let us first reproduce the commutator argument we used on $\mathbb{R}^2$. If $B = \arctan(x)\partial_x$, we have

$$[\widetilde{\Delta}, B] = 2 \langle x \rangle^{-2} \partial_x^2 - 2x \langle x \rangle^{-4} \partial_x + 2A'A^{-3} \arctan(x)\partial_\theta^2 + V_1' \arctan(x).$$

Now

$$iB - (iB)^* = i[\arctan(x), \partial_x]$$

is L^2 bounded, so

$$\begin{aligned} 0 &= \int_0^T \int \overline{u(\arctan(x)D_x(D_t - \widetilde{\Delta})u)}dx d\theta\, dt \\ &= \int_0^T \int \arctan(x)D_x u\overline{((D_t - \widetilde{\Delta})u)}dx d\theta\, dt \\ &\quad + \int_0^T \int (iB - (iB)^*)u\overline{((D_t - \widetilde{\Delta})u)}dx d\theta\, dt \\ &= i \langle \arctan(x)D_x u, u\rangle |_0^T + \int_0^T \left\langle (D_t - \widetilde{\Delta})i^{-1}Bu, u\right\rangle dt. \end{aligned}$$

Hence, using the notation $P = D_t - \widetilde{\Delta}$,

$$\begin{aligned} 0 &= 2i \operatorname{Im} \int_0^T \left\langle i^{-1}BPu, u\right\rangle dt \\ &= \int_0^T \left\langle i^{-1}BPu, u\right\rangle dt - \int_0^T \left\langle u, i^{-1}BPu\right\rangle dt \\ &= \int_0^T \left\langle [i^{-1}B, P]u, u\right\rangle dt - i \langle \arctan(x)D_x u, u\rangle |_0^T, \end{aligned}$$

or

$$\int_0^T \left\langle [B, -\widetilde{\Delta}]u, u\right\rangle dt = - \langle \arctan(x)D_x u, u\rangle |_0^T,$$

since B does not depend on t. By writing $\partial_x = \langle D_x\rangle^{1/2}\langle D_x\rangle^{-1/2}\partial_x$, and using energy estimates, we can control the right hand side by $\|u_0\|^2_{H^{1/2}}$. The left hand side is computed as above:

$$\begin{aligned}&\int_0^T \left\langle [B,-\widetilde{\Delta}]u,u\right\rangle dt\\ &= \int_0^T \left\langle (2\langle x\rangle^{-2}\partial_x^2 - 2x\langle x\rangle^{-4}\partial_x + 2A'A^{-3}\arctan(x)\partial_\theta^2\right.\\ &\qquad \left. + V_1'\arctan(x))u,u\right\rangle dt.\end{aligned}$$

Using the energy estimates,

$$\begin{aligned}&\left|\int_0^T \left\langle (-2x\langle x\rangle^{-4}\partial_x + V_1'\arctan(x))u,u\right\rangle dt\right| \leqslant CT\sup_{0\leqslant t\leqslant T}\|u(t)\|^2_{H^{1/2}}\\ &\leqslant C_T\|u_0\|^2_{H^{1/2}}.\end{aligned}\tag{2.3}$$

Integrating by parts in x and θ and adding the lower order terms into the right hand side as in (2.3) yields the estimate

$$\int_0^T (\|\langle x\rangle^{-1}\partial_x u\|^2_{L^2} + \|\sqrt{A'A^{-3}\arctan(x)}\partial_\theta u\|^2_{L^2})\,dt \leqslant C\|u_0\|^2_{H^{1/2}}.$$

We observe that

$$A'A^{-3}\arctan(x) = \arctan(x)x^{2m-1}(1+x^{2m})^{-1/m-1}$$

is even, non-negative, bounded below by $C|x|^{2m}$ for $|x|\leqslant 1$ and $C'|x|^{-3}$ for $|x|\geqslant 1$. Hence

$$|x|^{2m}\langle x\rangle^{-2m-3} \leqslant CA'A^{-3}\arctan(x),$$

and hence,

$$\left\langle |x|^{2m}\langle x\rangle^{-2m-3}\partial_\theta u,\partial_\theta u\right\rangle \leqslant C\left\langle A'A^{-3}\arctan(x)\partial_\theta u,\partial_\theta u\right\rangle$$

plus terms which can be absorbed into the energy, so up to lower order terms,

$$\||x|^m\langle x\rangle^{-m-3/2}\partial_\theta u\| \leqslant C\|\sqrt{A'A^{-3}\arctan(x)}\partial_\theta u\|.$$

Hence we have the estimate

$$\int_0^T (\|\langle x\rangle^{-1}\partial_x u\|^2_{L^2} + \||x|^m\langle x\rangle^{-m-3/2}\partial_\theta u\|^2_{L^2})\,dt \leqslant C\|u_0\|^2_{H^{1/2}}.\tag{2.4}$$

2.6. **Frequency decomposition and the estimate near $x=0$.** We split u into $u = u_{\mathrm{hi}} + u_{\mathrm{lo}}$, where u_{lo} has frequencies where the angular derivatives are controlled by the transversal ∂_x derivatives. The commutator method implies perfect local smoothing, even for u_{hi} away from $x=0$. In the interest of space, we have suppressed the details of this frequency decomposition. Let us now try to estimate u_{hi} near $x=0$, or more generally a solution to $(D_t+P_k)u=0$ microlocalized near $(0,0)$. For some $0\leqslant r\leqslant 1/2$ to be determined, let $F(t)$ be defined by

$$F(t)g = \chi(x)\psi(D_x/k)k^r e^{-itP_k}g,$$

where e^{-itP_k} is the Schrödinger propagator. Our goal is to determine for what values of r we have a mapping $F: L^2_x \to L^2([0,T])L^2_x$, since then

$$\|k^{1-r}F(t)u_0\|_{L^2([0,T]);L^2} \leqslant C\|k^{1-r}u_0\|_{L^2}\tag{2.5}$$

is the desired local smoothing estimate. We have such a mapping if and only if $FF^* : L^2L^2 \to L^2L^2$. We compute

$$FF^*f(x,t) = \psi(D_x/k)\chi(x)k^{2r}\int_0^T e^{i(t-s)P_k}\chi(x)\psi(D_x/k)f(x,s)ds,$$

and it suffices to estimate $\|FF^*f\|_{L^2L^2} \leqslant C\|f\|_{L^2L^2}$. We write $FF^*f(x,t) = \psi\chi(v_1+v_2)$, where

$$v_1 = k^{2r}\int_0^t e^{i(t-s)P_k}\chi(x)\psi(D_x/k)f(x,s)ds,$$

and

$$v_2 = k^{2r}\int_t^T e^{i(t-s)P_k}\chi(x)\psi(D_x/k)f(x,s)ds,$$

so that

$$(D_t+P_k)v_j = \pm ik^{2r}\chi\psi f,$$

and it suffices to estimate

$$\|\psi\chi v_j\|_{L^2L^2} \leqslant C\|f\|_{L^2L^2}.$$

Since the Fourier transform in time is an L^2 isometry, it suffices to estimate

$$\|\psi\chi\hat{v}_j\|_{L^2L^2} \leqslant C\|\hat{f}\|_{L^2L^2},$$

but this is the same as estimating

$$\|\psi\chi k^{2r}(\tau\pm i0+P_k)^{-1}\chi\psi\|_{L^2_x\to L^2_x} \leqslant C.$$

Let us factor out the k^2 in P_k to get the operator

$$k^{-2r}(\tau\pm i0+P_k) = k^{2(1-r)}(-z\pm i0+k^{-2}D_x^2+A^{-2}(x)+k^{-2}V_1(x))$$

for $-z=\tau k^{-2}$, and if we let $h=k^{-1}$, we are left with the task of finding r so that

$$\|\psi(hD_x)\chi(x)(-z\pm i0+(hD_x)^2+V)^{-1}\chi(x)\psi(hD_x)\|_{L^2\to L^2} \leqslant Ch^{-2(1-r)},$$

where $V = A^{-2}(x)+h^2V_1(x)$. Let

$$\widetilde{Q} = (hD_x)^2+V-z.$$

We observe that the cutoff $\psi(hD_x)\chi(x)$ shows we only need to estimate this for z in a bounded interval near $z=1$. Indeed, $\psi\chi$ cuts off to a neighbourhood of $(0,0)$, and $V(0)=1$, so for $|z-1|$ sufficiently large, we have elliptic regularity. The cutoff estimate on $\widetilde{Q}$ is the content of the following Proposition, which is proved in the next subsection.

Proposition 2.3. *Let $\varphi\in\Phi^0$ have wavefront set sufficiently close to $(0,0)$. Then for each $\epsilon>0$ sufficiently small, there exists a constant $C>0$ such that*

$$\|\varphi(\widetilde{Q}\pm i0)^{-1}\varphi\|_{L^2\to L^2} \leqslant Ch^{-2m/m+1},\ \ z\in[1-\epsilon,1+\epsilon].$$

With Proposition 2.3 in hand, we observe

$$\|\psi\chi k^{2r}(\tau\pm i0+P_k)^{-1}\chi\psi\|_{L^2_x\to L^2_x} \leqslant C$$

holds if

$$k^{2(r-1)} = k^{-2m/(m+1)},$$

or

$$r = \frac{1}{m+1}.$$

From (2.5), this implies Proposition 2.2 (see also [3, Theorem 1]).

2.7. **Proof of Proposition 2.3.** The technique of proof is to prove an invertibility estimate microlocally near $(0,0)$ in Lemma 2.4 below. The proof of the microlocal invertibility estimate proceeds through several steps. First, we rescale the principal symbol of $\widetilde{Q}$ to introduce a calculus of two parameters. We then quantize in the second parameter which eventually will be fixed as a constant in the problem. This technique has been used in [2,5,20,21].

The main estimate that glues the rest of the estimates together is a microlocal invertibility estimate for the operator $\widetilde{Q}$.

Lemma 2.4. *For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(T^*\mathbb{R})$ have compact support in $\{|(x,\xi)| \leqslant \epsilon\}$. Then there exists $C_\epsilon > 0$ such that*

$$\|\widetilde{Q}\varphi^w u\| \geqslant C_\epsilon h^{2m/(m+1)}\|\varphi^w u\|, \quad z \in [1-\epsilon, 1+\epsilon]. \tag{2.6}$$

2.8. **Proof of Lemma 2.4.** By virtue of the cutoff φ^w, to begin we are working microlocally in $\{|(x,\xi)| \leqslant \epsilon\}$. We observe that since $2m/(m+1) < 2$, if we can show the estimate (2.6) for $Q_1 = \widetilde{Q} - h^2 V_1$, the estimate follows also for $\widetilde{Q}$. Let

$$q_1 = \xi^2 + A^{-2} - z$$

be the principal symbol of Q_1. The function $A^{-2} = (1 + x^{2m})^{-1/m}$ is analytic near $x = 0$, and since $|x| \leqslant \epsilon$ is small, we expand A^{-2} in a Taylor series about $x = 0$ and write

$$q_1 = \xi^2 - \frac{1}{m}x^{2m}(1 + a(x)) - z_1,$$

where $z_1 = z - 1 \in [-\epsilon, \epsilon]$, and $a(x) = \mathcal{O}(x^{2m})$.

The Hamilton vector field H associated to the symbol q_1 is given by

$$\mathsf{H} = 2\xi\partial_x + (2x^{2m-1} + \mathcal{O}(x^{4m-1}))\partial_\xi.$$

We will consider a commutant localizing in this region and singular at the origin in a controlled way: we introduce new variables

$$\Xi = \frac{\xi}{(h/\tilde{h})^{m\alpha}}, \quad X = \frac{x}{(h/\tilde{h})^{\alpha}},$$

with

$$\alpha = \frac{1}{m+1}.$$

(When we wish to be more precise below, we will explicitly use the map $(x,\xi) = \mathcal{B}(X,\Xi)$ in this coordinate change; for the moment, we simply abuse notation.) As $m\alpha + \alpha = 1$, we note that quantizations of symbolic functions of X, Ξ lie in the pseudodifferential calculus, hence the symbol of the composition of two such operators depends *globally* on the symbols of the two operators. It is in order to cope with this issue that we employ the two parameter calculus.

We remark that in the new "blown-up" coordinates Ξ, X,

$$\mathsf{H} = (h/\tilde{h})^{\frac{m-1}{m+1}}\left(\Xi\partial_X + X^{2m-1}\partial_\Xi + \mathcal{O}((h/\tilde{h})^{2m\alpha}X^{2m})\partial_\Xi\right) \tag{2.7}$$

Now fix a small $\epsilon_0 > 0$ and set

$$\Lambda(s) = \int_0^s \langle s' \rangle^{-1-\epsilon_0}\, ds';$$

Λ is of course a symbol of order 0, with $\Lambda(s) \sim s$ near $s = 0$.

We introduce the singular symbol

$$a(x,\xi;h) = \Lambda(\Xi)\Lambda(X)\chi(x)\chi(\xi) = \Lambda(\xi/(h/\tilde{h})^{m\alpha})\Lambda(x/(h/\tilde{h})^{\alpha})\chi(x)\chi(\xi),$$

where $\chi(s)$ is a cutoff function equal to 1 for $|s| < \delta_1$ and 0 for $s > 2\delta_1$ (δ_1 will be chosen shortly). Then a is bounded, and a 0 symbol in X, Ξ :

$$\left|\partial_X^\alpha \partial_\Xi^\beta a\right| \leqslant C_{\alpha,\beta}.$$

(Recall that $x = (h/\tilde{h})^\alpha X$ and $\xi = (h/\tilde{h})^{m\alpha}\Xi$.) Using (2.7), it is simple to compute (2.8)

$$\begin{aligned}
\mathsf{H}(a) =&(h/\tilde{h})^{\frac{m-1}{m+1}}\chi(x)\chi(\xi)\big(\Lambda(\Xi)\langle X\rangle^{-1-\epsilon_0}\Xi \\
&+ X^{2m-1}\langle\Xi\rangle^{-1-\epsilon_0}\Lambda(X)(1+\mathcal{O}(x^{2m}))\big) + r \\
=&(h/\tilde{h})^{\frac{m-1}{m+1}}\chi(x)\chi(\xi)\Big((h/\tilde{h})^{-m\alpha}\xi\Lambda(\xi/(h/\tilde{h})^{m\alpha})\Big\langle x/(h/\tilde{h})^{\alpha}\Big\rangle^{-1-\epsilon_0} \\
&+ (h/\tilde{h})^{-(2m+1)\alpha}x^{2m-1}\Lambda(x/(h/\tilde{h})^{\alpha})\Big\langle \xi/(h/\tilde{h})^{m\alpha}\Big\rangle^{-1-\epsilon_0}(1+\mathcal{O}(x^{2m}))\Big) + r \\
\equiv&(h/\tilde{h})^{\frac{m-1}{m+1}} g + r
\end{aligned}$$

with

$$\operatorname{supp} r \subset \{|x| > \delta_1\} \cup \{|\xi| > \delta_1\}$$

(r comes from terms involving derivatives of $\chi(x)\chi(\xi)$). Note that near $X = \Xi = 0$, since $\Lambda(s) \sim s$ for $s \sim 0$, the term

$$g = \Lambda(\Xi)\langle X\rangle^{-1-\epsilon_0}\Xi + \langle\Xi\rangle^{-1-\epsilon_0}\Lambda(X)X^{2m-1}(1+\mathcal{O}(x^{2m})) \tag{2.9}$$

in $\mathsf{H}(a)$ is bounded below by a multiple of $\Xi^2 + X^{2m}$. Provided δ_1 is chosen small enough (so we can absorb the $\mathcal{O}(x^{2m})$ error term), g is in fact strictly positive away from $X = \Xi = 0$, while in the region $|(X,\Xi)| \geqslant 1$, we find that since $\operatorname{sgn}\Lambda(s) = \operatorname{sgn}(s)$, when $|\Xi| \geqslant \max(|X|^{1+\epsilon_0}, 1)$ then

$$g \geqslant \Lambda(\Xi)\langle X\rangle^{-1-\epsilon_0}\Xi \gtrsim \frac{|\Xi|}{\langle\Xi\rangle} \geqslant C > 0,$$

while for $|X|^{1+\epsilon_0} \geqslant \max(|\Xi|, 1)$, we have (providing $\delta_1 \ll 1$)

$$g \geqslant (1/2)\langle\Xi\rangle^{-1-\epsilon_0}\Lambda(X)X^{2m-1} \gtrsim |X|^{-2(1+\epsilon_0)}|X|^{2m-1} \geqslant C > 0,$$

provided $2(1+\epsilon_0) < 2m-1$. Thus, since the larger of $|\Xi|$ and $|X|^{1+\epsilon_0}$ is assuredly greater than 1 in the region of interest, we have in fact shown that

$$g \geqslant C > 0 \quad \text{in } \{\Xi^2 + X^2 > 1\}.$$

Thus, we find

$$\mathsf{H}(a) = (h/\tilde{h})^{\frac{m-1}{m+1}} g + r$$

with

$$r = \mathcal{O}_{\mathcal{S}_{\alpha,\beta}}((h/\tilde{h})^{(m-1)/(m+1)}((h/\tilde{h})^{\alpha}|\Xi| + (h/\tilde{h})^{\beta}|X^{2m-1}|)$$

supported as above and

$$g(X,\Xi;h) = \begin{cases} c(\Xi^2 + X^{2m})(1+r_2), & \Xi^2 + X^2 \leqslant 1 \\ b, & \Xi^2 + X^2 \geqslant 1, \end{cases}$$

where $c > 0$ is a constant,$r_2 = \mathcal{O}_{\mathcal{S}_{\alpha,\beta}}(\delta_1)$, and $b > 0$ is elliptic.

We will require a positivity result dealing with operators satisfying estimates of this type.

Lemma 2.5. *Let a real-valued symbol $\tilde{g}(x,\xi;h)$ satisfy*

$$\tilde{g}(x,\xi;h) = \begin{cases} c(\xi^2 + x^{2m})(1 + r_2), & \xi^2 + x^2 \leqslant 1 \\ b, & \xi^2 + x^2 \geqslant 1, \end{cases}$$

where $c > 0$ is constant, $r_2 = \mathcal{O}_{\mathcal{S}_{\alpha,\beta}}(\delta_1)$, and $b > 0$ is elliptic. Then there exists $c_0 > 0$ such that

$$\langle \operatorname{Op}_h^w(\tilde{g})u, u\rangle \geqslant c_0 h^{2m/(m+1)} \|u\|^2$$

for h sufficiently small.

We now employ this result to estimate $\operatorname{Op}_h^w(\mathsf{H}(a))$.

Lemma 2.6. *For $\tilde{h} > 0$ sufficiently small, there exists $c > 0$ such that $\operatorname{Op}_h^w(g) > c\tilde{h}^{2m/(m+1)}$, uniformly as $h \downarrow 0$, where g is given by (2.9).*

Proof. Note that we have written g as a function of X, Ξ, so in changing variables to x, ξ we are tacitly employing the blowdown map $\mathcal{B}$. In particular, we are interested in estimating $\operatorname{Op}_h^w(g \circ \mathcal{B}^{-1})$ from below. By conjugating by the rescaling operator $T_{h,\tilde{h}}$, we have the change in calculus formula:

$$\operatorname{Op}_{\tilde{h}}^w(g) T_{h,\tilde{h}} u = T_{h,\tilde{h}} \operatorname{Op}_h^w(g \circ \mathcal{B}^{-1}) u,$$

hence

$$\langle \operatorname{Op}_h^w(g \circ \mathcal{B}^{-1})u, u\rangle = \left\langle T_{h,\tilde{h}} \operatorname{Op}_{\tilde{h}}^w(g) T_{h,\tilde{h}} u, u \right\rangle \geqslant c\tilde{h}^{2m/(m+1)} \|u\|^2$$

for $\tilde{h}$ sufficiently small, by unitarity of $T_{h,\tilde{h}}$ and Lemma 2.5, with $\tilde{h}$ replacing h. This establishes the Lemma. ☐

The lower order terms from the proof of Lemma 2.4 are handled in [9], which we leave out here in the interest of exposition.

We are now able to prove the resolvent estimate Lemma 2.4. Let $v = \varphi^w u$, with φ chosen to have support inside the set where $\chi(x)\chi(\xi) = 1$; thus the terms r and r_3 above are supported away from the support of φ. Then Lemma 2.6 and some careful analysis of the error terms yield

$$\begin{aligned} i\langle [Q_1 - z, a^w] v, v\rangle &= h\langle \operatorname{Op}_h^w(\mathsf{H}(a))v, v\rangle + \langle \operatorname{Op}_h^w(e_2)u, u\rangle \\ &= h(h/\tilde{h})^{(m-1)/(m+1)} \langle \operatorname{Op}_h^w(g)v, v\rangle + \langle \operatorname{Op}_h^w(e_2)u, u\rangle \\ &= h^{2m/(m+1)} \big(\tilde{h}^{-(m-1)/(m+1)} + \mathcal{O}(\tilde{h}^{-(m-3)/(m+1)})\big) \langle \operatorname{Op}_h^w(g)v, v\rangle \\ &\geqslant C h^{2m/(m+1)} \tilde{h} \|v\|^2, \end{aligned}$$

for $\tilde{h}$ sufficiently small. On the other hand, we certainly have

$$\big|\langle [Q_1 - z, a^w] v, v\rangle\big| \leqslant C \|(Q_1 - z)v\| \|v\|,$$

hence the desired bound follows once we fix $\tilde{h} > 0$. ☐

3. Results from [8] and [6]

The techniques in [9] are robust enough to apply in several different situations. In [8], J. Metcalfe and the author study a new kind of trapping called "inflection-transmission" trapping. The basic idea is that, instead of a maximum of the effective potential at $x = 0$, we allow an inflection point, say, at $x = 1$. The induced symbol looks like $\xi^2 - (x-1)^{2m_1+1}$ for $m_1 \geqslant 1$ a natural number.

That is, in the notation above, we consider a surface of revolution with generating curve given by the following construction. Let m_1 and m_2 be positive integers, and set

$$A^2(x) = 1 + \int_0^x y^{2m_1-1}(y-1)^{2m_2}/(1+y^2)^{m_1+m_2-1}dy.$$

As the integrand in the last term

$$x^{2m_1-1}(x-1)^{2m_2}/(1+x^2)^{m_1+m_2-1} \sim \begin{cases} x^{2m_1-1}, & x \sim 0, \\ (x-1)^{2m_2}/2^{m_1+m_2-1} & x \sim 1, \\ x, & |x| \to \infty, \end{cases}$$

we notice that

$$A^2(x) \sim \begin{cases} 1 + x^{2m_1}, & x \sim 0, \\ C_1 + c_2(x-1)^{2m_2+1} & x \sim 1, \\ x^2, & |x| \to \infty. \end{cases} \tag{3.1}$$

Here $C_1 > 1$ and $c_2 < 1$ are constants which are easily computed but inessential, except for their relative sizes compared to 1. As will be clear in the sequel, the specific structure of A is inessential and only the location and nature of the critical points and behavior at infinity matter.

In [9], the main idea from the technical commutator argument near the critical point works because

$$\{x\xi, \xi^2 - x^{2m}\} \sim \xi^2 + x^{2m}.$$

That is, differentiating x^{2m} produces an *odd* power of x near 0, but the additional x in $x\xi$ multiplies to give an *even* power, which is non-negative definite. Due to the odd power in $(x-1)^{2m_2+1}$, differentiating makes an even power, so our commutant must result in an elliptic multiple of this. It is very interesting that this is accomplished by considering exactly the same commutant. That is, since x is elliptic near $x = 1$, we observe

$$\{x\xi, \xi^2 - (x-1)^{2m_2+1}\} \sim \xi^2 + x(x-1)^{2m_2},$$

which is non-negative near $x = 1$.

In this case, the polynomial power in the loss in local smoothing is different. The main result of [8] is the following theorem:

Theorem 2 (Local Smoothing). *Suppose X is as above with $m_1, m_2 \geqslant 1$ and assume u solves*

$$\begin{cases} (D_t - \Delta)u = 0 \text{ in } \mathbb{R} \times X, \\ u|_{t=0} = u_0 \in H^s \end{cases}$$

for some $s > 0$ sufficiently large. Then for any $T < \infty$, there exists a constant $C_T > 0$ such that

$$\int_0^T \left(\| \langle x \rangle^{-1} \partial_x u\|^2_{L^2(dVol)} + \| \langle x \rangle^{-3/2} \partial_\theta u\|^2_{L^2(dVol)} \right) dt$$
$$\leqslant C_T \left(\| \langle D_\theta \rangle^{\beta(m_1,m_2)} u_0\|^2_{L^2(dVol)} + \| \langle D_x \rangle^{1/2} u_0\|^2_{L^2(dVol)} \right),$$

where

$$\beta(m_1, m_2) = \max\left(\frac{m_1}{m_1+1}, \frac{2m_2+1}{2m_2+3} \right). \tag{3.2}$$

Moreover this estimate is sharp, in the sense that no polynomial improvement in regularity is true.

3.1. **Further developments.** After the papers [8, 9], the author continued his study of degenerate trapping. Having handled the cases of finitely degenerate maxima and inflection points, the next logical step is to consider infinitely degenerate critical points, and indeed even segments of critical points. In these cases, wave packets near the trapping are so slow to escape to infinity that no local smoothing is expected. Indeed, Theorems 1 and 2 give no smoothing in the limit as $m \to \infty$. Hence only a microlocal resolvent estimate is expected. In [6], the author develops techniques to handle these infinitely degenerate cases. Essentially, a calculus is developed around adding an h-dependent finitely degenerate "bump" to the effective potential. Then a similar argument to that in [8, 9] is employed to get a good estimate on the perturbed operator. In comparing the perturbed operator to the original operator, there is an unavoidable loss in h for the microlocal invertibility estimate. In Lemma 2.4, the microlocal resolvent estimate has an upper bound on the order of $h^{-2m/(m+1)}$. It is important to note that the exponent $2m/(m+1) < 2$ for all m. In the infinitely degenerate case, the upper bound is $C_\epsilon h^{-2-\epsilon}$ for each $\epsilon > 0$. In other words, there is a gap between the estimates for finitely degenerate trapping and infinitely degenerate trapping.

4. Related results on compact domains

In this section, we give one last very simple application of the grand-daddy commutator to eigenfunctions in planar domains.

Let $\Omega \subset \mathbb{R}^2$ be a compact domain with piecewise smooth boundary. Consider the Dirichlet eigenfunction problem:

$$\begin{cases} -h^2 \Delta u = u, \\ u|_{\partial\Omega} = 0, \end{cases}$$

and assume that $\|u\| = 1$. Let $X = (x+m)\partial_x + (y+n)\partial_y$ with m, n parameters independent of (x, y) and h. Observe this is just a constant coefficient perturbation of the radial vector field. Hence we still have $[-h^2\Delta, X] = -2h^2\Delta$. Hence we can integrate by parts:

$$\begin{aligned} 2 &= -2\int_\Omega (h^2\Delta u)\bar{u}dV \\ &= \int_\Omega ([-h^2\Delta - 1, X]u)\bar{u}dV \\ &= \int_\Omega ((-h^2\Delta - 1)Xu)\bar{u}dV \\ &= \int_{\partial\Omega} (hXu)h\partial_\nu \bar{u}dS. \end{aligned}$$

The point is that this commutator method gives information about how iterior behaviour of the eigenfunctions yields information about the Neumann data. One can, of course, jazz up the vector field X with spatial cutoffs or microlocal cutoffs. Along with J. Toth, the author is working to apply this method not just to the boundary of Ω, but also to interior hypersurfaces.

In order to discover ways to apply this to planar domains, the author considered first partially rectangular domains, which is a rectangle with "wings" attached to two opposite sides. In turn, the author considered simpler and simpler domains until discovering a remarkable theorem [7]: The mass of the (semi-classical) Neumann data on each side of a triangle is equal to the length of the side divided by the area of the triangle. This is not an asymptotic, but an exact formula.

Let us briefly summarize the proof, since it is so simple. Given a triangle T with sides of length a, b, c, if we compute the boundary terms, we find that the contribution of the $x\partial_x$ cancels with the contribution of $y\partial_y$ on each side. Then there are explicit constants $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3$ such that

$$\begin{aligned} 2 &= \int_{\partial T} (hXu)h\partial_\nu \bar{u} dS \\ &= (A_1 m + A_2 n + A_3) \int_a |h\partial_\nu u|^2 + (B_1 m + B_2 n + B_3) \int_b |h\partial_\nu u|^2 \\ &\quad + (C_1 m + C_2 n + C_3) \int_c |h\partial_\nu u|^2. \end{aligned} \tag{4.1}$$

Let

$$I_a = \int_a |h\partial_\nu u|^2 dS$$

and similarly for I_b and I_c.

The constants in (4.1) are explicitly computed in terms of the side lengths a, b, c. If we set $m = n = 0$, we get one linear equation for the I_a, I_b, I_c. Differentiating (4.1) with respect to m and n yields two more independent linear equations for the quantities I_a, I_b, and I_c. When we solve these equations, we get

$$I_a = \frac{a}{\text{Area}(T)}, \ I_b = \frac{b}{\text{Area}(T)}, \ I_c = \frac{c}{\text{Area}(T)}.$$

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E-mail address: hans@math.unc.edu

Department of Mathematics, UNC-Chapel Hill, CB#3250 Phillips Hall, Chapel Hill, NC 27599